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Coloring *p*4*m* with Two, Four, and Six Colors

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Abstract

The relationship of equivalence for symmetric colorings *vs* equivalence for color groups is illustrated for the group *p*4*m*. The seven subgroups of index 2, the 31 subgroups of index 4, and the 12 subgroups of index 6 are listed. The relationsip of these 50 subgroups and colorings to the corresponding 20 color-group types is explained. Each coloring is sketched in a figure.

Introduction

Much attention has been devoted in recent years to color symmetry and the classification of color groups. For example, see Senechal (1979), Harker (1981), Jarratt & Schwarzenberger (1980, 1981), MacDonald & Street (1978*a*, *b*), Nabonnand & Billiet (1983), and Wieting (1982). A recent article by Schwarzenberger (1984) surveys the whole field and its history. However, the question of equivalence has been a point of some controversy. The usual approach is to classify color groups using the following definition of equivalence: a color group is a pair (G, H) consisting of a symmetry group or crystallographic group G together with a subgroup H of finite index n: if H and H' are two such subgroups then (G, H) *and* (G, H') *are said to be equivalent if there is an affine transformation f normalizing G with* $fHf^{-1} = H'$ [see Schwarzenberger (1980), p. 39]. We will say that they belong to the same *color-group type.*

The author (Roth, 1982), however, has emphasized a different point of view, namely that the original design or structure and its symmetry group G are given and one wishes to classify the symmetric colorings of that structure. *Two symmetric colorings of a given structure are equivalent if a relabelling of the colors transforms the first coloring to the second coloring.* Two different subgroups will then give inequivalent colorings (see Roth, 1982, theorem 1.3). Another reference that explores this point of view is a recent paper of Senechal (1983). To list the symmetric colorings with n colors for a structure with a given symmetry group G one must then list all the subgroups of index n.

We feel that much information is lost if one only lists the color-group types and that for many purposes it is important to take into account the different subgroups (and colorings) that correspond to one color-group type. This paper is intended as a case study to illustrate the situation for *p*4*m* with two, four and six colors (no such colorings exist for $n=3$ or 5) by listing all subgroups of index 2, 4 and 6 and displaying the colorings of one particular design for these cases. A comparison is made with the related color groups that are classified by the definition of equivalence cited above and the various situations that occur are analyzed. While *p*4*m* is of course just one of the 17 planar crystallographic groups it will serve as a typical example of the various types of anomalies that can occur for any symmetry group. For *p*4*m* using two, four and six colors there are 50 colorings corresponding to 20 color-group types.

The theoretical background for this work has been developed in detail by the author (Roth, 1982) and the notation and approach of that paper are used throughout; we quickly summarize it here. Note that we compose symmetry operations from left to right so that **ba** means first **b** then **a**. The design or structure must first be partitioned into 'fundamental regions', which form an orbit under the action of G and have the property that for any two regions A_i, A_j there is a unique symmetry operation g in G such that g maps A_i onto A_j. One chooses one of the regions Ω to be the 'starting region'; then each region A is of the form Ωg for a unique g and is labelled by the element g. If H is a subgroup of index n in G then one uses the right coset decomposition of G: $G = Hx_1 \cup Hx_2 \cup \ldots \cup Hx_n$. The regions labelled by the elements of the coset Hx_i are then colored by the color i for $i = 1, 2, \ldots, n$. We generally let x_1 be the identity element **e** of G so that the regions corresponding to the subgroup H would be colored by color 1. This gives a symmetric coloring of the design; an element g maps precisely the set of regions colored i onto the regions colored j if $Hx_ig = Hx_j$. The correspondence from the set of subgroups to the set of colorings will depend on the choice of 'starting region' Ω (which is the region labelled by the identity element **e**); however, under any such choice the collection of colorings associated with a set of conjugate subgroups remains the same.

There are two ways that the color groups (G, H) and (G, H') could be equivalent: they may be

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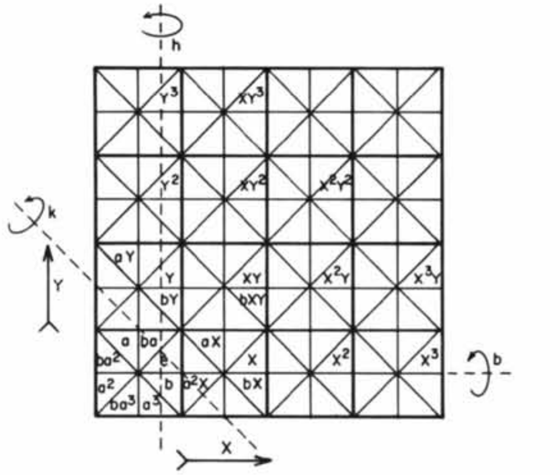

Fig. 1. The fundamental regions are labelled by the elements of the group $p4m$ using the generators, which consist of the translations **x** and **y**, the rotation **a** and the reflection **b**. **e** denotes the identity. The colorings for two and four colors will be illustrated using only the lower left 3 by 3 array (Figs. 2–7) while the illustrations with six colors will use the 4 by 4 array (Figs. 8, 9). **k** and **h** are reflections in the axes shown and do not belong to the original group $G = p4m$.

equivalent under an 'inner' automorphism induced by conjugation by an element of the symmetry group G itself, in which case H and H' will necessarily be conjugate subgroups; or they may be equivalent under an 'outer' automorphism induced by conjugating by an element not in G (this is a slight abuse of the usual group-theoretical use of the word 'outer automorphism' since we will not insist that it not be equal to some inner automorphism). In certain cases both situations occur simultaneously (see the discussion of H_{37} and H_{38} later). We should note that in the important paper of van der Waerden & Burckhardt (1961), where the basic approach to coloring used here originated, the color groups (G, H) and (G, H') are essentially considered 'equivalent' only when inner automorphisms are involved, and this more restrictive definition is still used for certain applications by some authors [see Litvin, Kotzev & Birman (1982), for example]. Now in the case of $G = p4m$, a design having G as symmetry group has two sets of inequivalent fourfold centers. A geometric transformation normalizing G either leaves these two sets invariant (in which case it belongs to G itself) or interchanges the two sets: thus G is of index two in its normalizer, which also happens to be a group of type $p4m$. It is useful to single out a particular element of this larger group, namely the reflection **k** in the diagonal axis bisecting the starting fundamental region; see Fig. 1. This element, which together with G will generate the normalizer of G, plays a special role as will be explained below.

We have chosen as the design to be colored the familiar rectangular grid that divides the plane into

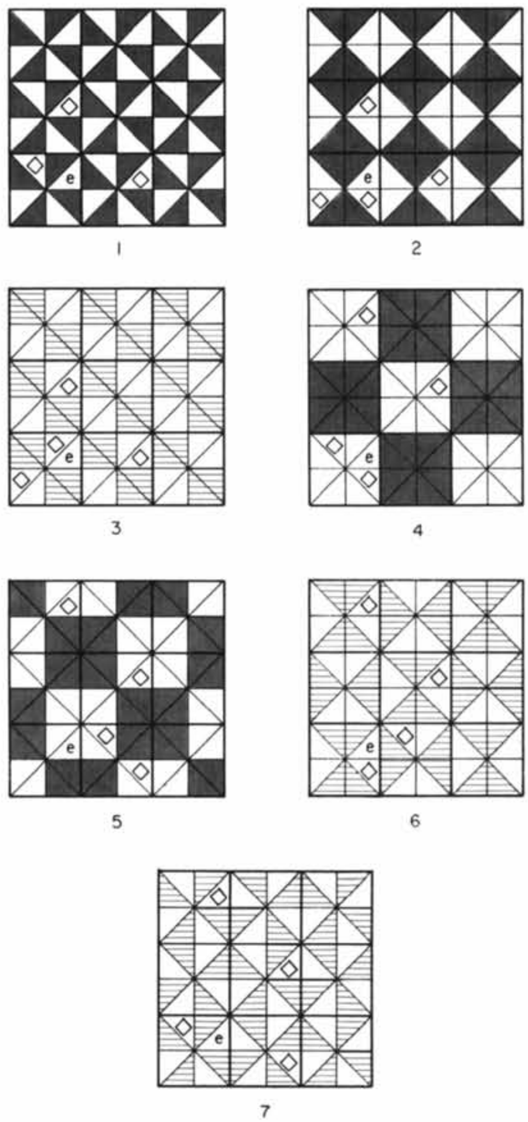

Fig. 2. The seven colorings for $p4m$ using two colors. The starting region is labelled **e** and the regions corresponding to the generators given in the table are each marked by a diamond here and in the other figures.

an array of squares. The fundamental domains are then triangles, each square being divided into eight such triangles. See Fig. 1. Note that the basic original design that is being colored consists of the larger squares formed by the heavier lines, not the collection of triangles (nor the finer lines) that are added in preparation for the colorings. This design fills the plane completely. Other designs could also be used for $p4m$ and if they consist of smaller motifs that do not fill the plane then the appearance of the colorings would be different. If the fundamental domains are chosen as large as possible, so as to fill out the plane, then they will necessarily be the triangular regions

Table 1. *Table of colorings*

Coloring number	Subgroup H_m	Conjugates	Color-group type	Subgroup type	n	μ	Δ	Fig.
1	$\langle a, x, y\rangle$	Normal	CG1	*p4*	2	2	1	2
2	$\langle a^2, b, x, y\rangle$	Normal	CG2	*pmm*	2	2	1	2
3	$\langle a^2, ba, x, y\rangle$	Normal	CG3	*cmm*	2	2	1	2
4	$\langle a, b, xy, y^2\rangle$	Normal	CG4	*p4m*	2	1	2	2
5	$\langle ax, bx, xy, y^2\rangle$	Normal						
6	$\langle ax, b, xy, y^2\rangle$	Normal	CG5	*p4g*	2	1	2	2
7	$\langle a, bx, xy, y^2\rangle$	Normal						
8	$\langle a^2, x, y\rangle$	Normal	CG6	*p2*	4	4	1	3
9	$\langle a^2, ba, xy, y^2\rangle$	Normal	CG7	*pmm*	4	2	2	3
10	$\langle a^2, bax, xy, y^2\rangle$	Normal	CG8	*pgg*	4	2	2	3
11	$\langle a^2, b, xy, y^2\rangle$	Normal	CG9	*cmm*	4	2	2	3
12	$\langle a^2, bx, xy, y^2\rangle$	Normal						
13	$\langle a, xy, y^2\rangle$	Normal	CG10	*p4*	4	2	2	3
14	$\langle ax, xy, y^2\rangle$	Normal						
15	$\langle a, b, x^2, y^2\rangle$	Conj. to H_{16}	CG11	*p4m*	4	1	4	4
16	$\langle axy, b, x^2, y^2\rangle$	Conj. to H_{15}						
17	$\langle ax, by, x^2, y^2\rangle$	Conj. to H_{18}						
18	$\langle ay, by, x^2, y^2\rangle$	Conj. to H_{17}						
19	$\langle ay, bx, x^2, y^2\rangle$	Conj. to H_{20}	CG12	*p4g*	4	1	4	5
20	$\langle ax, bx, x^2, y^2\rangle$	Conj. to H_{19}						
21	$\langle axy, bxy, x^2, y^2\rangle$	Conj. to H_{22}						
22	$\langle a, bxy, x^2, y^2\rangle$	Conj. to H_{21}						
23	$\langle a^2, b, x, y^2\rangle$	Conj. to H_{24}	CG13	*pmm*	4	2	2	5
24	$\langle a^2, b, x^2, y\rangle$	Conj. to H_{23}						
25	$\langle a^2x, b, x^2, y\rangle$	Conj. to H_{26}						
26	$\langle a^2y, by, x, y^2\rangle$	Conj. to H_{25}						
27	$\langle a^2y, b, x, y^2\rangle$	Conj. to H_{28}	CG14	*pmg*	4	2	2	6
28	$\langle a^2x, bx, y, x^2\rangle$	Conj. to H_{27}						
29	$\langle a^2, bx, y, x^2\rangle$	Conj. to H_{30}						
30	$\langle a^2, by, x, y^2\rangle$	Conj. to H_{29}						
31	$\langle ba, x, y\rangle$	Conj. to H_{32}	CG15	*cm*	4	4	1	6
32	$\langle ba^3, x, y\rangle$	Conj. to H_{31}						
33	$\langle a^2x, b, xy, y^2\rangle$	Conj. to H_{34}	CG16	*cmm*	4	2	2	7
34	$\langle a^2x, bx, xy, y^2\rangle$	Conj. to H_{33}						
35	$\langle a^2x, ba, xy, y^2\rangle$	Conj. to H_{36}	CG17	*pmg*	4	2	2	7
36	$\langle a^2x, ba^3, xy, y^2\rangle$	Conj. to H_{35}						
37	$\langle b, x, y\rangle$	Conj. to H_{38}	CG18	*pm*	4	4	1	7
38	$\langle ba^2, x, y\rangle$	Conj. to H_{37}						
39	$\langle a^2, b, x, y^3\rangle$	All six	CG19	*pmm*	6	2	3	8
40	$\langle a^2x, b, x^3, y\rangle$	subgroups						
41	$\langle a^2, b, x^3, y\rangle$	H_{39}–H_{44}						
42	$\langle a^2y, by, x, y^3\rangle$	are						
43	$\langle a^2x^2, b, x^3, y\rangle$	conjugate						
44	$\langle a^2y^2, by^2, x, y^3\rangle$							
45	$\langle a^2y, ba^3, xy, y^3\rangle$	All six	CG20	*cmm*	6	2	3	8
46	$\langle a^2x, bax, xy, y^3\rangle$	subgroups						
47	$\langle a^2y^2, ba, xy^2, y^3\rangle$	H_{45}–H_{50}						
48	$\langle a^2, ba, xy^2, y^3\rangle$	are						
49	$\langle a^2, ba, xy, y^3\rangle$	conjugate						
50	$\langle a^2y, ba, xy^2, y^3\rangle$							

that we are using, because the edges of these triangles are the axes of reflections that generate the group; so the colorings would appear as in our figures. However, for those symmetry groups that are not generated by such reflections (such as *p*4 or *p*4*g*) there is no such canonical choice for the fundamental regions and the appearance of the colorings could vary considerably. This does not affect the analysis of the colorings that we are giving, except with respect to the question of a certain apparent similarity of the patterns that arises in some cases; for example, see colorings 4 and 5 in Fig. 2. This phenomenon was perhaps noted first by MacDonald & Street (1976). If colors are added to a design one could ignore the original design (here the rectangular grid of squares) and simply consider the new configuration consisting of the pattern of colors that has been laid down on the design. It is these 'color patterns' that may then appear to be the same or very similar; in the example of colorings 4 and 5, they are congruent under a symmetry operation that is not in the original group *G*. Because this similarity depends so much on the choice of the original design and the fundamental domains we feel that this aspect should not distract one from our emphasis on the basic classification by equivalence of colorings.

Further study of the apparent 'similarity' of inequivalent colorings is important but we feel the proper analysis should be done in the framework of partial color symmetry (using a larger group contain-

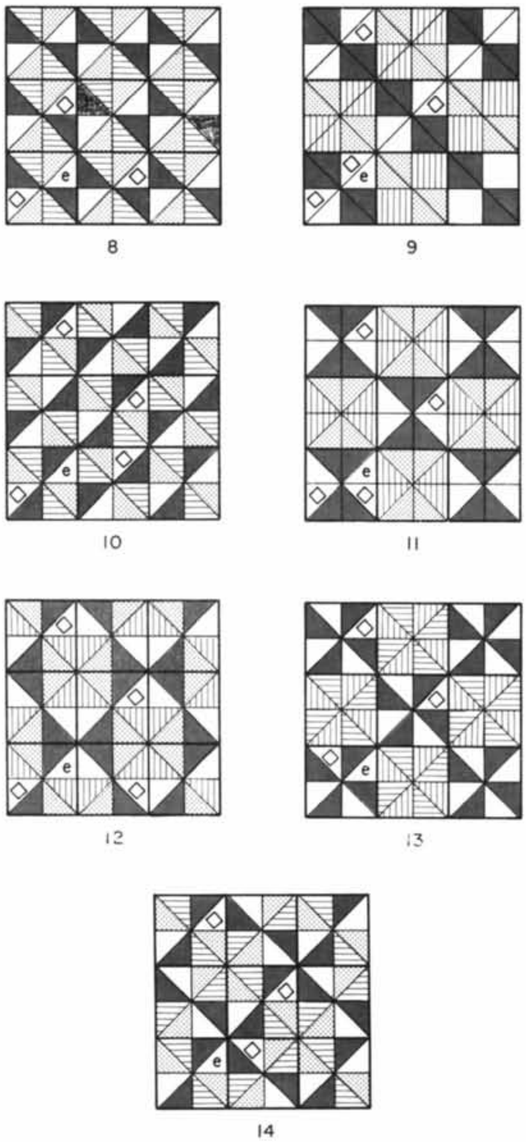

Fig. 3. Colorings 8-14.

ing G) as is formulated in § 4 of the author's 1982 paper, where they were called 'equivalent color patterns'. From the discussion in that paper [see discussion at top of p. 290 in Roth (1982)], it follows that when H is sent to H' under the automorphism induced by the reflection **k** (that is if $H' = k^{-1}Hk$) then the color patterns are 'equivalent', being transformed one to the other under the geometric symmetry operation **k**; for short we will say that the colorings are 'k congruent'. In this case, since **k** normalizes G, the color groups involved are equivalent. Other symmetry operations that normalize G may also have the same effect on the color patterns (and in fact in the figures illustrated it is often easier to spot a translation that effects the transformation); however, theorem

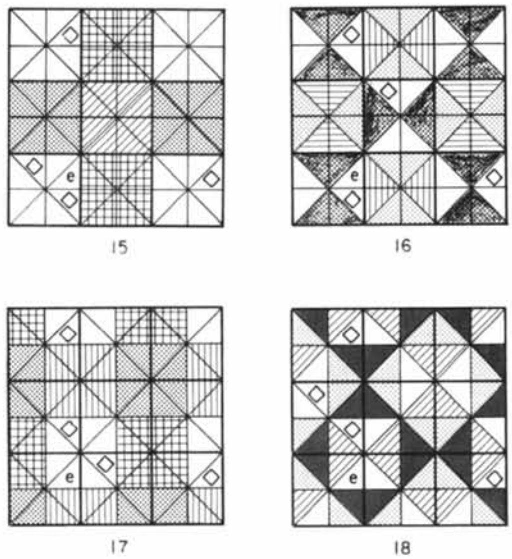

Fig. 4. Colorings 15-18 corresponding to the color-group type CG11.

4.3 of Roth (1982) shows that if the color patterns are 'equivalent' under a symmetry operation that normalizes G then they are actually k congruent. However, it will also be seen that sometimes the color patterns are 'equivalent' under a symmetry operation that does not normalize G and the color groups might not in fact be equivalent. Thus the fact that this phenomenon often arises in connection with inequivalent colorings belonging to the same color-group type should not be used as a justification for listing only the color-group types; the classification by equivalence of colorings that we are stressing in this paper is also important.

In generating $p4m$, we use **x** and **y** as the generating translations, horizontal and vertical, respectively; **a** is a rotation 90° counterclockwise around the center of a fixed square and **b** is a reflection in the horizontal axis through that center. See Fig. 1. Then $p4m$ is generated by these four elements; *i.e.* $p4m = \langle \mathbf{a}, \mathbf{b}, \mathbf{x}, \mathbf{y} \rangle$. The colorings are numbered $m = 1$ to 50 and for each m the coloring is determined by the corresponding subgroup H_m.

Senechal (1979) has described a program for finding the subgroups of the plane crystallographic groups [see also Senechal (1980) for higher dimensions; and see Sayari, Billiet & Zarrouk (1978) for a listing of the subgroups of the two-dimensional space groups]. Following the notation of Senechal (1979) let T denote the group of translations (here $T = \langle \mathbf{x}, \mathbf{y} \rangle$). S denotes the point group; in this case S may be identified with the dihedral group $\langle \mathbf{a}, \mathbf{b} \rangle$. For any subgroup H of finite index, let T' denote its translational subgroup. Then $H/T' \simeq S'$, a subgroup of S. To find all subgroups H one must first choose

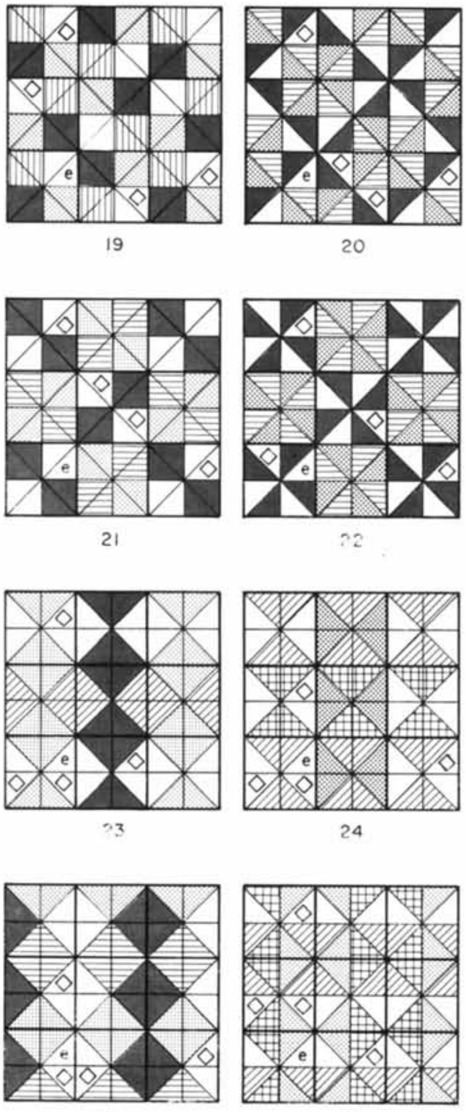

Fig. 5. Colorings 19–22 for color-group type CG12 and colorings 23–36 for color-group type CG13.

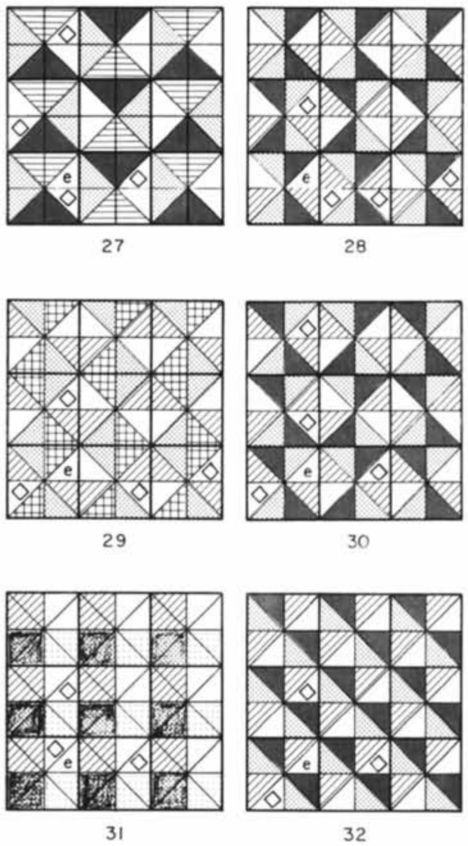

Fig. 6. Colorings 27–30 for color-group type CG14 and colorings 31–32 for color-group type CG15.

a subgroup T' and a subgroup S' under which T' is invariant; to check this one may pick coset representatives for G/T corresponding to the elements of S (in this example they can be the elements of $\langle \mathbf{a}, \mathbf{b} \rangle$ themselves), then find the ones that correspond to the generators of the subgroups S' (call these $\{c, d\}$, say, if there are two of them), and operate with them on T' by conjugation. With T' and S', representatives for the cosets of H modulo T' must be selected appropriately (choose coset representatives for T modulo T', combine them with c and d and check that modulo T' they do generate a factor group corresponding to S'); see theorem 1.2 of Senechal (1979). Then the number of colors is $n = [G:H] = \mu\Delta$, where $\mu = [S:S']$ and $\Delta = [T:T']$. In the table of colorings (Table 1) we list the subgroups by generators putting the generators of T' last and it will be clear what S' and T' are in each case. For example, subgroup $H_{12} = \langle \mathbf{a}^2, \mathbf{bx}, \mathbf{xy}, \mathbf{y}^2 \rangle$; here $T' = \langle \mathbf{xy}, \mathbf{y}^2 \rangle$ is of index $\Delta = 2$ in T while $S' = \langle \mathbf{a}^2, \mathbf{b} \rangle$ is of index $\mu = 2$ in S. The 20 color-group types are denoted CG1, . . . , CG20.

Discussion of the colorings

For the case of two colors there are seven subgroups of index 2 in $G = p4m$ [see also Nabonnand & Billiet (1983)], so there are seven colorings as shown in Fig. 2. These correspond to five color-group types. H_4 and H_5 yield equivalent color groups; *i.e.* they both correspond to the color-group type CG4. It is also seen from the figure that the color patterns are k congruent but the colorings are inequivalent since the relationship of the colors to the original figure is clearly different. Similarly, the pair of colorings 6 and 7 correspond to color-group CG5 and the colorings are also k congruent. Subgroups H_1, H_2, and H_3 each correspond to a different color-group type. It is also seen that in these three cases an application of the

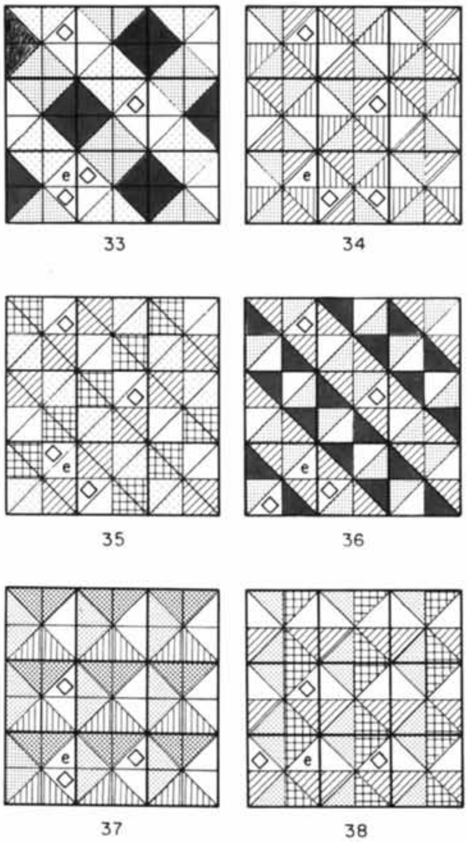

Fig. 7. Colorings 33-38.

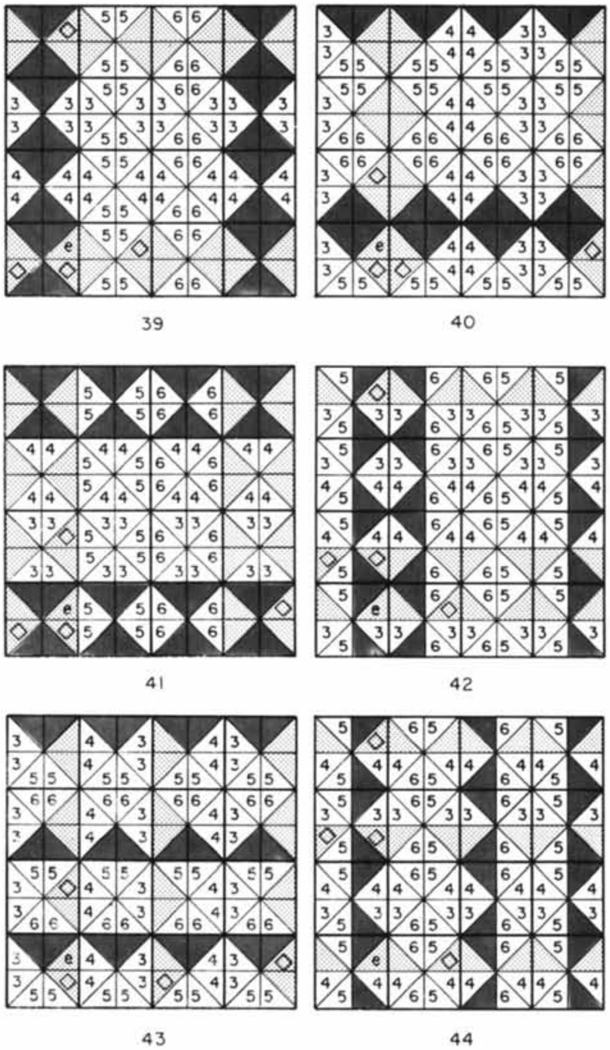

Fig. 8. Colorings 39-44 for color-group type CG19.

reflection **k** maps the portion representing the subgroup H onto itself. Since the seven subgroups of index 2 are of course all normal, the phenomenon of conjugate subgroups is not illustrated here.

There are 31 subgroups of index 4. The subgroups H_8 through H_{14} are all normal subgroups of index 4 (see Fig. 3). Again it is seen that H_8, H_9, and H_{10} each afford different color-group types while the pairs H_{11}, H_{12} and H_{13}, H_{14} each correspond to color-group types CG9 and CG10 respectively. As is evident from the figure, the reflection **k** leaves each of the subgroups H_8, H_9, H_{10} invariant while interchanging the pair H_{11} and H_{12} and interchanging H_{13} and H_{14}.

In the case of the 14 normal subgroups just discussed the fact that those colorings corresponding to the same color-group type also yield 'equivalent color patterns', as discussed earlier, would appear to give some justification to the use of *color groups* under the usual equivalence as opposed to *colorings*. However, the situation is more complicated as will be seen in further examples. Compare the four colorings 15, 16, 17, 18 corresponding to color-group type CG11 (see Fig. 4). H_{15} and H_{16} are conjugate subgroups as are H_{17} and H_{18}. It is clear that the colorings 15 and 16 are quite different from each other as are 17 and 18. However, colorings 15 and 17 are similar (though, we emphasize again, not equivalent) since the reflection **k** transforms one to the other (up to a relabelling of the colors; note that in the illustrated figures the 'colors' used may or may not happen to match up under this transformation) and 16 and 18 are also k congruent. The four subgroups form two conjugate pairs; any two of these four subgroups would be 'conjugate' if we allow elements from the larger group $\langle \mathbf{a}, \mathbf{b}, \mathbf{x}, \mathbf{y}, \mathbf{k} \rangle$. For example, $x^{-1}H_{15}x = H_{16}$, $x^{-1}H_{17}x = H_{18}$, $k^{-1}H_{15}k = H_{17}$, $k^{-1}H_{16}k = H_{18}$ and hence $(xk)^{-1}H_{15}(xk) = H_{18}$. The situation with each of the color-group types CG12, CG13, CG14 is similar to that of CG11: each is afforded by four subgroups consisting of two conjugate pairs and related by the reflection **k** as above. See colorings 19-30 (Figs. 5 and 6).

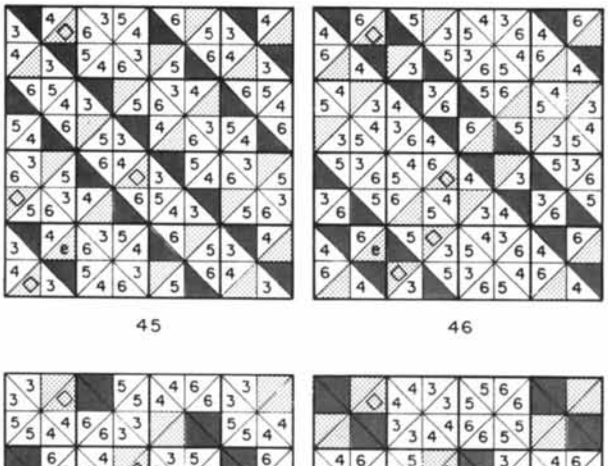

Fig. 9. Colorings 45–50 for color-group type CG20.

Subgroups H_{31} and H_{32} are conjugate and each is invariant under **k**; they correspond to the one-color-group type CG15 (see Fig. 6). A similar situation holds for the pair H_{33} and H_{34} and the pair H_{35} and H_{36}. The pair H_{37} and H_{38} is a bit more unusual; they are conjugate subgroups and they may also be transformed one to the other under conjugation by the reflection **k**, *i.e.* their colorings are k congruent (see Fig. 7).

A further anomaly is apparent if one compares colorings 9 and 35; the patterns are similar and they are 'congruent' under the reflection **h** (see Fig. 1), which does not normalize G. It is clear that the associated color-group types CG7 and CG17, respectively, are different. In fact, H_9 is normal in G while H_{35} is not normal in G. This example was discussed in detail as example 3 of § 4 in Roth (1982).

There are twelve subgroups of index 6 in $p4m$. They correspond to the two color-group types CG19 and CG20. The six subgroups H_{39}–H_{44} (color-group type CG19, see Fig. 8) are all conjugate. In addition, colorings 39 and 40 are k congruent (this is similar to the situation for the pair 37 and 38 described above) as are the pair 41, 42 and the pair 43, 44.

The six conjugate subgroups H_{45}–H_{50} corresponding to color-group type CG20 (see Fig. 9) show a different story. Colorings 45 and 46 are each invariant under **k** while 47 and 48 are interchanged by **k**. Colorings 49 and 50 are each invariant under **k**. However, in this last case the color patterns are congruent under the reflection **h**. This is similar to the situation of subgroups H_9 and H_{35} (discussed earlier) except that here the two subgroups also happen to be conjugate and hence to correspond to the same color-group type, whereas the other example involved distinct color-group types. Thus here, in the case of colorings 49 and 50, the fact that the color patterns are equivalent and that the color-group type is the same is merely a coincidence, because they are congruent under the symmetry operation **h**, which does not normalize the group G.

References

HARKER, D. (1981). *Acta Cryst.* **A37**, 286–292.

JARRATT, J. D. & SCHWARZENBERGER, R. L. E. (1980). *Acta Cryst.* **A36**, 884–888.

JARRATT, J. D. & SCHWARZENBERGER, R. L. E. (1981). *Util. Math.* **19**, 295–303.

LITVIN, D. B., KOTSEV, J. N. & BIRMAN, J. L. (1982). *Phys. Rev. B*, **26**, 6947–6970.

MACDONALD, S. O. & STREET, A. P. (1976). *Lect. Notes Math.* **560**, 149–157.

MACDONALD, S. O. & STREET, A. P. (1978*a*). *Lect. Notes Math.* **686**, 210–222.

MACDONALD, S. O. & STREET, A. P. (1978*b*). *Util. Math.* **13**, 271–292.

NABONNAND, P. & BILLIET, Y. (1983). Abstr. Eighth Eur. Crystallogr. Meet., p. 291.

ROTH, R. L. (1982). *Discrete Math.* **38**, 273–296.

SAYARI, A., BILLIET, Y. & ZARROUK, H. (1978). *Acta Cryst.* **A34**, 553–555.

SCHWARZENBERGER, R. L. E. (1980). *N-Dimensional Crystallography.* San Francisco, London, Melbourne: Pitman.

SCHWARZENBERGER, R. L. E. (1984). *Bull. London Math. Soc.* **16**, 209–240.

SENECHAL, M. (1979). *Discrete Appl. Math.* **1**, 51–73.

SENECHAL, M. (1980). *Acta Cryst.* **A36**, 845–850.

SENECHAL, M. (1983). *Acta Cryst.* **A39**, 505–511.

WAERDEN, B. L. VAN DER & BURCKHARDT, J. J. (1961). *Z. Kristallogr.* **115**, 231–234.

WIETING, T. W. (1982). *Mathematical Theory of Chromatic Plane Ornaments.* New York: Marcel Dekker.